# Theoretical 3: Solution

## Physics of Spin

### Part A. Larmor Precession

1. From the two equations given in the text, we obtain the relation

$$\frac{d\boldsymbol{\mu}}{dt} = -\gamma\boldsymbol{\mu} \times \mathbf{B}. \tag{1}$$

Taking the dot product of eq (1). with $\boldsymbol{\mu}$, we can prove that

$$\begin{aligned}\boldsymbol{\mu} \cdot \frac{d\boldsymbol{\mu}}{dt} &= -\gamma\boldsymbol{\mu} \cdot (\boldsymbol{\mu} \times \mathbf{B}), \\ \frac{d|\boldsymbol{\mu}|^2}{dt} &= 0, \\ \mu = |\boldsymbol{\mu}| &= \text{const.}\end{aligned} \tag{2}$$

Taking the dot product of eq. (1) with $\mathbf{B}$, we also prove that

$$\begin{aligned}\mathbf{B} \cdot \frac{d\boldsymbol{\mu}}{dt} &= -\gamma\mathbf{B} \cdot (\boldsymbol{\mu} \times \mathbf{B}), \\ \mathbf{B} \cdot \frac{d\boldsymbol{\mu}}{dt} &= 0, \\ \mathbf{B} \cdot \boldsymbol{\mu} &= \text{const.}\end{aligned} \tag{3}$$

An acute reader will notice that our master equation in (1) is identical to the equation of motion for a charged particle in a magnetic field

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{v} \times \mathbf{B}. \tag{4}$$

Hence, the same argument for a charged particle in magnetic field can be applied in this case.

2. For a magnetic moment making an angle of $\phi$ with $\mathbf{B}$,

$$\begin{aligned}\frac{d\boldsymbol{\mu}}{dt} &= -\gamma\boldsymbol{\mu} \times \mathbf{B}, \\ |\boldsymbol{\mu}| \sin\phi \frac{d\theta}{dt} &= \gamma |\boldsymbol{\mu}| B_0 \sin\phi, \\ \omega_0 = \frac{d\theta}{dt} &= \gamma B_0.\end{aligned} \tag{5}$$

### Part B. Rotating frame

1. Using the relation given in the text, it is easily shown that

$$\begin{aligned}\left(\frac{d\boldsymbol{\mu}}{dt}\right)_{\text{rot}} &= \left(\frac{d\boldsymbol{\mu}}{dt}\right)_{\text{lab}} - \boldsymbol{\omega} \times \boldsymbol{\mu} \\ &= -\gamma\boldsymbol{\mu} \times \mathbf{B} - \omega\mathbf{k}' \times \boldsymbol{\mu} \\ &= -\gamma\boldsymbol{\mu} \times \left(\mathbf{B} - \frac{\omega}{\gamma}\mathbf{k}'\right) \\ &= -\gamma\boldsymbol{\mu} \times \mathbf{B}_{\text{eff}}.\end{aligned} \tag{6}$$

Note that $\mathbf{k}$ is equal to $\mathbf{k}'$ as observed in the rotating frame.

2. The new precession frequency as viewed on the rotating frame $S'$ is

$$\begin{aligned} \vec{\Delta} &= (\omega_0 - \omega)\,\mathbf{k}', \\ \Delta &= \gamma B_0 - \omega. \end{aligned} \tag{7}$$

3. Since the magnetic field as viewed in the rotating frame is $\mathbf{B} = B_0\mathbf{k}' + b\mathbf{i}'$,

$$\mathbf{B}_{\text{eff}} = \mathbf{B} - \omega/\gamma\mathbf{k}' = \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b\mathbf{i}',$$

and

$$\begin{aligned} \Omega &= \gamma\,|\mathbf{B}_{\text{eff}}|\,, \\ &= \gamma\sqrt{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + b^2}. \end{aligned} \tag{8}$$

4. In this case, the effective magnetic field becomes

$$\begin{aligned} \mathbf{B}_{\text{eff}} &= \mathbf{B} - \omega/\gamma\mathbf{k}' \\ &= \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b(\cos 2\omega t\mathbf{i}' - \sin 2\omega t\mathbf{j}') \end{aligned} \tag{9}$$

which has a time average of $\overline{\mathbf{B}_{\text{eff}}} = \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}'$.

**Part C. Rabi oscillation**

1. The oscillating field can be considered as a superposition of two oppositely rotating field:

$$2b\cos\omega_0 t\mathbf{i} = b\left(\cos\omega_0 t\mathbf{i} + \sin\omega_0 t\mathbf{j}\right) + b\left(\cos\omega_0 t\mathbf{i} - \sin\omega_0 t\mathbf{j}\right),$$

which gives an effective field of (with $\omega = \omega_0 = \gamma B_0$):

$$\mathbf{B}_{\text{eff}} = \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b\mathbf{i}' + b(\cos 2\omega_0 t\mathbf{i}' - \sin 2\omega_0 t\mathbf{j}').$$

Since $\omega_0 \gg \gamma b$, the rotation of the term $b(\cos 2\omega_0 t\mathbf{i}' - \sin 2\omega_0 t\mathbf{j}')$ is so fast compared to the frequency $\gamma b$. This means that we can take the approximation

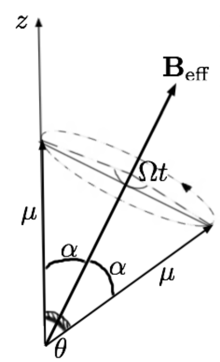
$$\mathbf{B}_{\text{eff}} \approx \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b\mathbf{i}' = b\mathbf{i}', \tag{10}$$

where the magnetic moment precesses with frequency $\Omega = \gamma b$.

As $\Omega = \gamma b \ll \omega_0$, the magnetic moment does not "feel" the rotating term $b\left(\cos 2\omega_0 t\mathbf{i}' - \sin 2\omega_0 t\mathbf{j}'\right)$ which averaged to zero.

2. Since the angle $\alpha$ that $\boldsymbol{\mu}$ makes with $\mathbf{B}_{\text{eff}}$ stays constant and $\boldsymbol{\mu}$ is initially oriented along the $z$ axis, $\alpha$ is also the angle between $\mathbf{B}_{\text{eff}}$ and the $z$ axis which is

$$\tan\alpha = \frac{b}{B_0 - \frac{\omega}{\gamma}}. \tag{11}$$




From the geometry of the system, we can show that ($\cos\theta = \mu_z/\mu$):

$$\begin{aligned}
2\mu\sin\frac{\theta}{2} &= 2\mu\sin\alpha\sin\frac{\Omega t}{2}, \\
\sin^2\frac{\theta}{2} &= \sin^2\alpha\sin^2\frac{\Omega t}{2}, \\
\frac{1-\cos\theta}{2} &= \sin^2\alpha\frac{1-\cos\Omega t}{2}, \\
\cos\theta &= 1-\sin^2\alpha+\sin^2\alpha\cos\Omega t, \\
\cos\theta &= \cos^2\alpha+\sin^2\alpha\cos\Omega t.
\end{aligned}$$

So, the projected magnetic moment along the $z$ axis is $\mu_z(t) = \mu\cos\theta$ and the magnetization is

$$M = N\mu_z = N\mu\left(\cos^2\alpha + \sin^2\alpha\cos\Omega t\right). \tag{12}$$

Note that the magnetization does not depend on the reference frame $S$ or $S'$ ($\mu_z$ has the same value viewed in both frames).

Taking $\omega = \omega_0 = \gamma B_0$, the angle $\alpha$ is $90^0$ and $M = N\mu\cos\Omega t$.

3. From the relations

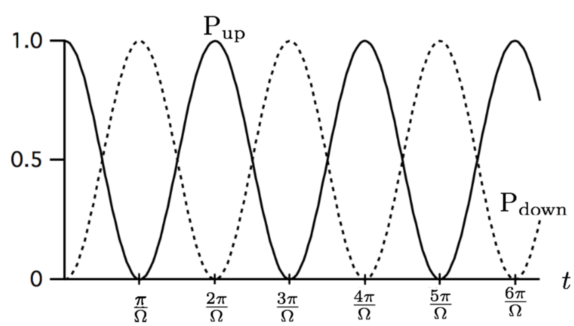
$$\begin{aligned}
P_\uparrow - P_\downarrow &= \frac{\mu_z}{\mu} = \cos\theta, \\
P_\uparrow + P_\downarrow &= 1,
\end{aligned}$$

we obtain the results ($\omega = \omega_0$)

$$\begin{aligned} P_\downarrow &= \frac{1-\cos\theta}{2} \\ &= \frac{1-\cos^2\alpha - \sin^2\alpha\cos\Omega t}{2} \\ &= \sin^2\alpha\frac{1-\cos\Omega t}{2} \\ &= \frac{b^2}{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + b^2}\sin^2\frac{\Omega t}{2} \\ &= \sin^2\frac{\Omega t}{2}, \end{aligned} \tag{13}$$

and

$$P_\uparrow = \frac{b^2}{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + b^2}\cos^2\frac{\Omega t}{2} = \cos^2\frac{\Omega t}{2}. \tag{14}$$




**Part D. Measurement incompatibility**

1. In the $x$ direction, the uncertainty in position due to the screen opening is $\Delta x$. According to the uncertainty principle, the atom momentum uncertainty $\Delta p_x$ is given by

   $$\Delta p_x \approx \frac{\hbar}{\Delta x},$$

   which translates into an uncertainty in the $x$ velocity of the atom,

   $$v_x \approx \frac{\hbar}{m\Delta x}.$$

   Consequently, during the time of flight $t$ of the atoms through the device, the uncertainty in the width of the beam will grow by an amount $\delta x$ given by

   $$\delta x = \Delta v_x t \approx \frac{\hbar}{m\Delta x}t.$$

So, the width of the beams is growing linearly in time. Meanwhile, the two beams are separating at a rate determined by the force $F_x$ and the separation between the beams after a time $t$ becomes

$$d_x = 2 \times \frac{1}{2}\frac{F_x}{m}t^2 = \frac{1}{m}\left|\mu_x\right| Ct^2.$$

In order to be able to distinguish which beam a particle belongs to, the separation of the two beams must be greater than the widths of the beams; otherwise the two beams will overlap and it will be impossible to know what the $x$ component of the atom spin is. Thus, the condition must be satisfied is

$$\begin{aligned} d_x &\gg \delta x, \\ \frac{1}{m}\left|\mu_x\right| Ct^2 &\gg \frac{\hbar}{m\Delta x}t, \\ \frac{1}{\hbar}\left|\mu_x\right| \Delta x Ct &\gg 1. \end{aligned} \tag{15}$$

2. As the atoms pass through the screen, the variation of magnetic field strength across the beam width experienced by the atoms is

$$\Delta B = \Delta x \frac{dB}{dx} = C\Delta x.$$

This means the atoms will precess at rates covering a range of values $\Delta\omega$ given by

$$\Delta\omega = \gamma\Delta B = \frac{\mu_z}{\hbar}\Delta B = \frac{\left|\mu_x\right|}{\hbar}C\Delta x,$$

and, if previous condition in measuring $\mu_x$ is satisfied,

$$\Delta\omega t \gg 1. \tag{16}$$

In other words, the spread in the angle $\Delta\omega t$ through which the magnetic moments precess is so large that the $z$ component of the spin is completely randomized or the measurement uncertainty is very large.