Mechanics of a Deformable Lattice (Total Marks : 20)

Here we study a deformable lattice hanging in gravity which acts as a deformable physical pendulum. It has only one degree of freedom, i.e. only one way to deform it and the configuration is fully described by an angle α. Such structures have been studied by famous physicist James Maxwell in 19^{th} century, and some surprising behaviors have been discovered recently.

As shown in the figure 1, N^2 identical triangular plates (red triangle) are freely hinged by identical rods and form an $N \times N$ lattice ($N > 1$). The joints at the vertices are denoted by small circles. The sides of the equilateral triangles and the rods have the same length l. The dashed lines in the figure represent four tubes; each tube confines N vertices (grey circles) on the edge and the N vertices can slide in the tube, i.e. the tube is like a sliding rail.

The four tubes are connected in a diamond shape with two angles fixed at 60° and another two angles at 120° as shown in Figure 1. Each plate has a uniform density with mass M, and the other parts of the system are massless. The configuration of the lattice is uniquely determined by the angle α, where $0^\circ \leq \alpha \leq 60^\circ$ (please see the examples of different angle α in Figure 1). The system is hung vertically like a "curtain" with the top tube fixed along the horizontal direction.

The coordinate system is shown in Figure 2. The zero level of the potential energy is defined at $y = 0$. A triangular plate is denoted by a pair of indices (m,n), where $m, n = 0, 1, 2, \cdots, N-1$ representing the order in the x and y directions respectively. A(m,n), B(m,n) and C(m,n) denote the positions of the 3 vertices of the triangle (m,n). The top-left vertex, A(0, 0) (the big black circle), is fixed.

The motion of the whole system is confined in the x-y plane. The moment of inertia of a uniform equilateral triangular plate about its center of mass is $I = Ml^2/12$. The free fall acceleration is g. Please use E_k and E_p to denote kinetic energy and potential energy respectively.

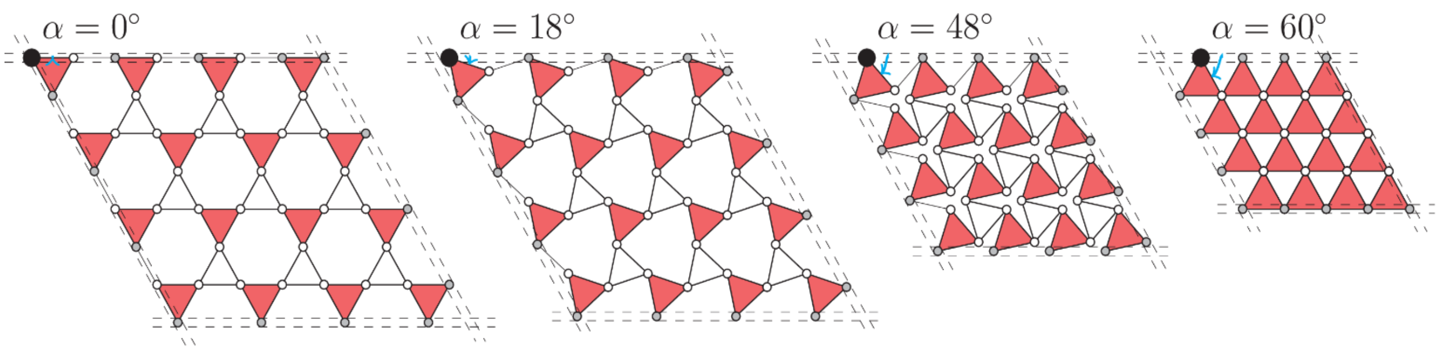

Figure 1

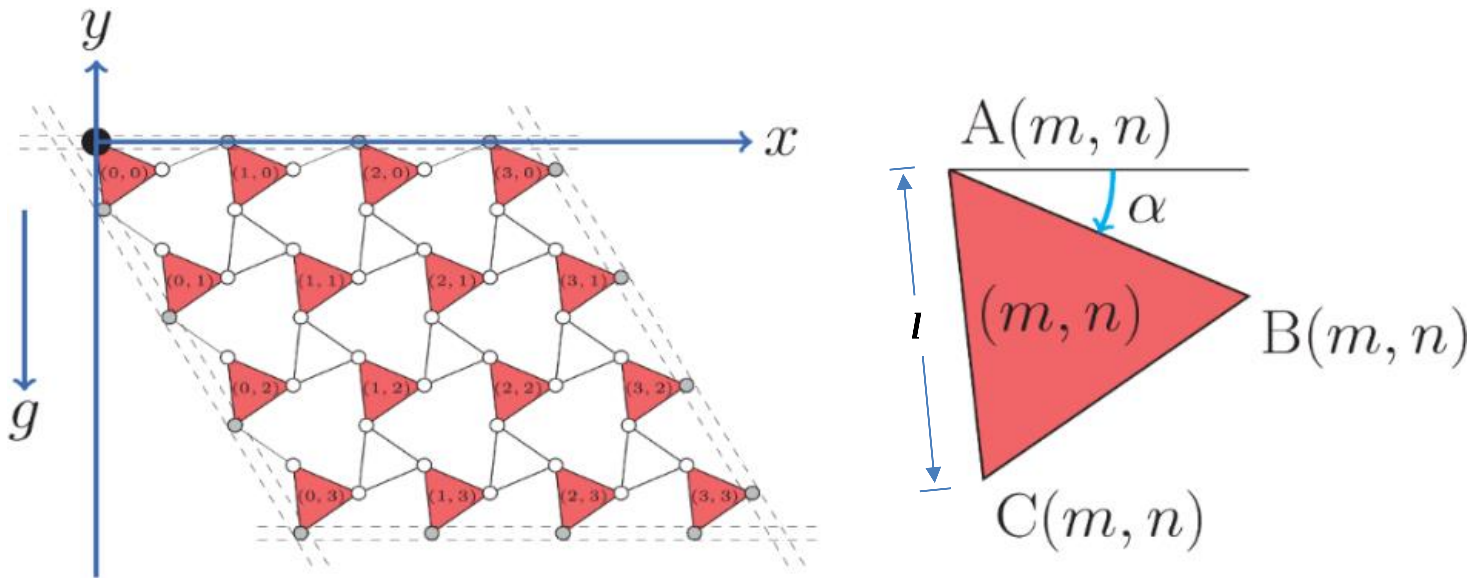

Figure 2

Section A: When N=2 (as shown in figure 3):

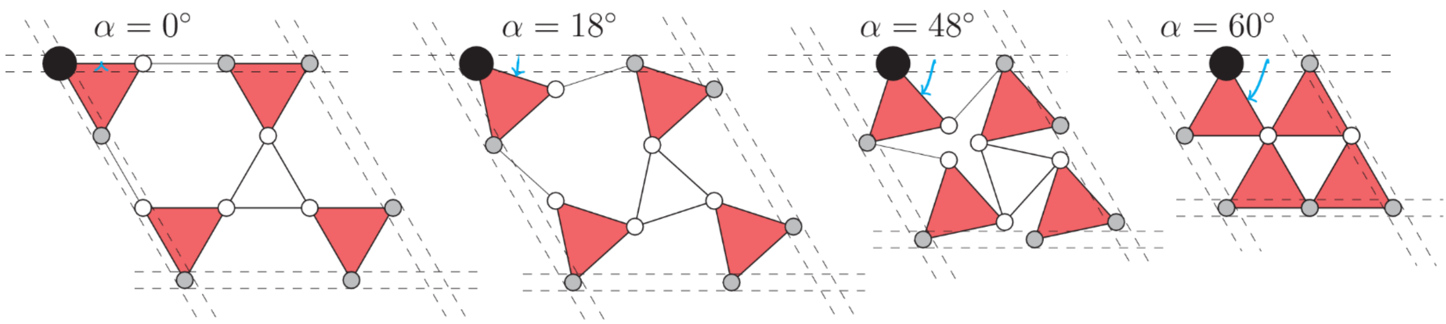

Figure 3

A1	What is the potential energy E_p of the system for a general angle α when N = 2?	**2 points**

A2	What is the equilibrium angle α_E of the system under gravity when N = 2?	**1 point**

A3	The system follows a simple harmonic oscillation under a small perturbation from equilibrium. Calculate the kinetic energy of this system in terms of $\Delta\dot{\alpha} \equiv \mathrm{d}(\Delta\alpha)/\mathrm{d}t$. Calculate the oscillation frequency f_E when N = 2.	**5 points**

Section B: For arbitrary N :

B1	What is the equilibrium angle α'_{E} under gravity when N is arbitrary?	**3 points**

B2	Consider the case when $N \to \infty$. Under a small perturbation of angle α, the change of potential energy of the system is $\Delta E_{\mathrm{p}} \propto N^{\gamma_1}$, the kinetic energy of the system is $E_{\mathrm{k}} \propto N^{\gamma_2}$, and the oscillation frequency is $f'_{\mathrm{E}} \propto N^{\gamma_3}$. Find the values of γ_1, γ_2 and γ_3.	**3 points**

Section C: A force is exerted on one of the $3N^2$ triangle vertices so that the system maintains at $\alpha_{\mathrm{m}} = 60°$.

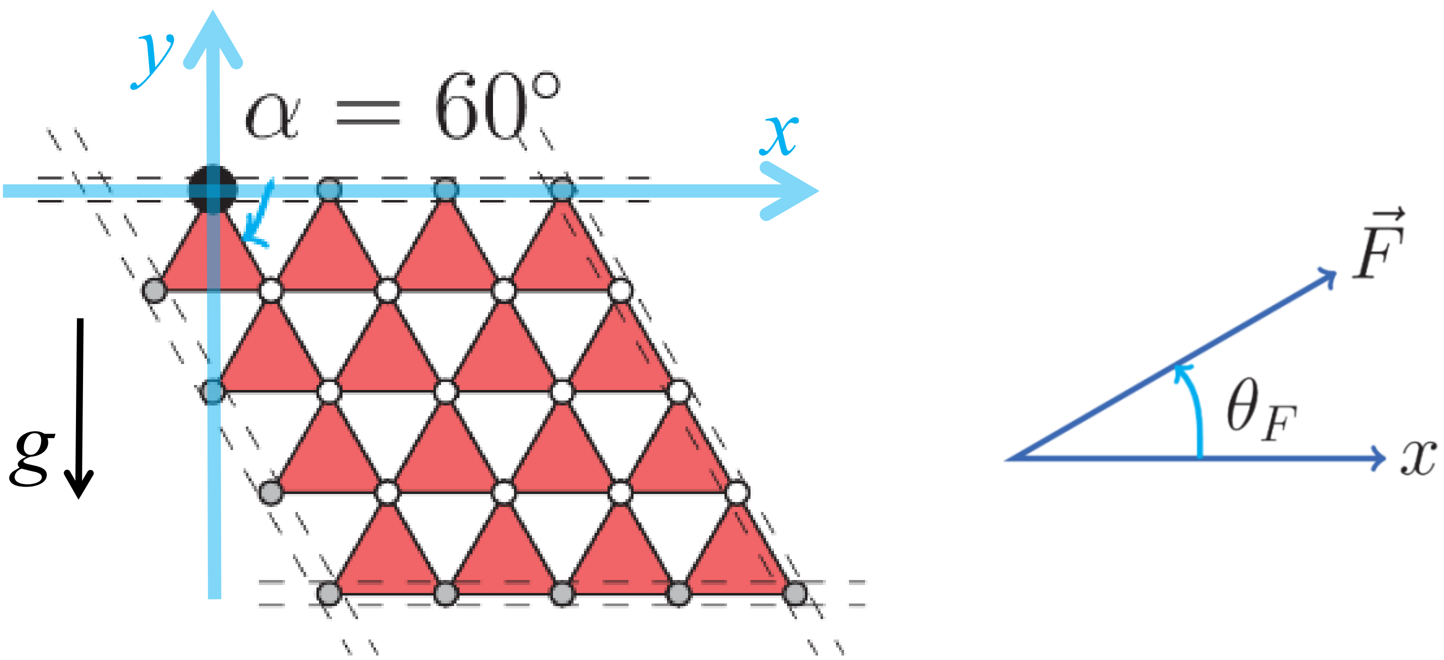

Figure 4

C1	Which vertex should we choose to minimize the magnitude of this force?	**1 point**

C2	What are the direction and magnitude of this minimum force? Describe the direction in terms of the angle θ_F defined in Figure 4.	**5 points**